A THEOREM ON THE STABILITY OF MOTION

(ODNA TEOREMA OB USTOICHIVOSTI DVIZHENIA)

PMM Vol.24, No.1, 1960, pp. 47-54

V.V. RUMIANTSEV
(Moscow)

(Received 14 November 1959)

1. Let us consider an arbitrary holonomic mechanical system, and let $q_1, \ldots, q_n$ be its independent Lagrangian coordinates, and $q_1', \ldots, q_n'$ be its generalized velocities. We shall assume that the equations of motion of the system have the particular solution

$$q_i = f_i(t) \qquad (i = 1, \ldots, n) \tag{1.1}$$

corresponding to the unperturbed motion.

Let t_0 be the initial instant of the time t, and q_{i0} and q_{i0}' be the initial values of the variable q_i and its time derivative q_i'.

Let for the unperturbed motion

$$q_{i0} = f_i(t_0), \qquad q_{i0}' = f_i'(t_0)$$

and for the perturbed motion

$$q_{i0} = f_i(t_0) + \varepsilon_i, \qquad q_{i0}' = f_i'(t_0) + \varepsilon_i'$$

where ϵ_i ϵ_i' are real constants designated as perturbations. The introduction of these constants defines completely the perturbed motion, because the forces acting on the system are assumed to remain unchanged [1].

Let the values of the coordinates q_i and of the generalized velocities q_i' for the perturbed motion be

$$q_i = f_i(t) + x_i, \qquad q_i' = f_i'(t) + x_{n+i}$$

where $x_j(t)$, $(j = 1, \ldots, 2n)$ are the variations of the variables q_i and q_i', which satisfy the equations of the perturbed motion

$$\frac{dx_j}{dt} = X_j(t, x_1, \ldots, x_{2n}) \qquad (j = 1, \ldots, 2n) \tag{1.2}$$

We shall assume that for every $t > t_0$ the functions $X_j(t, x_1, \ldots, x_{2n})$ can be expanded in convergent power series with integral exponents in the variables $x_1, \ldots, x_{2n}$, whose coefficients are real continuous functions of t, with $x_j(t, 0, \ldots, 0) = 0$.

We are interested in the stability of the unperturbed motion (1.1) with respect to certain real continuous functions $Q_1, \ldots, Q_k$ of the variables q_i, q_i' and the time t. For the unperturbed motion the functions Q_s after substituting in them $q_i = f_i(t)$ and $q_i' = f_i'(t)$ transform into certain known functions of the time $F_s(t)$, and for the perturbed motion they will transform into certain functions of the time t and of the perturbations ϵ_i, ϵ_i'. Considering the differences $y_s = Q_s = F_s$, Liapunov called the unperturbed motion (1.1) stable with respect to the quantities $Q_1, \ldots, Q_k$, if for all L_s, however small, there exist positive numbers E_i, E_i' such that for any ϵ_i, ϵ_i' satisfying the condition

$$|\varepsilon_i| \leqslant E_i, \qquad |\varepsilon_i'| \leqslant E_i'$$

and for any $t > t_0$, the following inequalities hold:

$$|y_s| < L_s \qquad (s = 1, \ldots, k)$$

We shall assume further [1] that to any set of real values of the perturbations ϵ_i, ϵ_i' numerically sufficiently small, there corresponds a certain set of real initial values y_{s0} of the variables y_s, and that we could always satisfy the inequality

$$y_{10}^2 + \cdots + y_{k0}^2 < A$$

for any positive value of A, however small, if the absolute values of the perturbations are not greater than sufficiently small non-zero numbers E_i, E_i'. And conversely, for given positive numbers E_i, E_i', however small, there exists such a positive number A, that to the quantity $y_{10}^2 + \ldots + y_{k0}^2 < A$ there corresponds one or several sets of real ϵ_i, ϵ_i' whose absolute values are smaller than E_i and E_i' respectively.

Since the variables y_s represent certain functions of the variables t, x_j, vanishing when all $x_j = 0$, $(j = 1, \ldots, 2n)$, then the region of variations of the real variables $t, x_1, \ldots, x_{2n}$

$$t \geqslant t_0, \qquad x_1^2 + \cdots + x_{2n}^2 \leqslant H \tag{1.3}$$

where t_0 and $H > 0$ are constants, will correspond to the region

$$t \geqslant t_0, \qquad y_1^2 + \cdots + y_k^2 \leqslant H_1 \tag{1.4}$$

of variations of the variables t, y_s, where $H_1 > 0$ is a constant.

We shall assume that (1.2) yields the first integral

$$\varphi(x_1, \ldots, x_{2n}, t) = \text{const} \tag{1.5}$$

where $\phi(x_1, \ldots, x_{2n}, t)$ is a real, continuous, bounded, single-valued function of its variables in the region (1.3) vanishing when all the variables x_j are zero. Suppose also that at all values of the variables t, x_j in the region (1.3) and at all corresponding values of the variables t, y_s in the region (1.4) the following inequality is satisfied:

$$\Phi(y_1, \ldots, y_k, t) \leqslant \varphi(x_1, \ldots, x_{2n}, t) \tag{1.6}$$

Here $\Phi(y_1, \ldots, y_k, t)$ is a real, continuous, bounded, single-valued function of its variables in the region (1.4), vanishing when all the y_s are zero. We can prove now the following theorem:

Theorem. If the differential equations of the perturbed motion (1.2) admit a first integral (1.5) and it is possible to find a positive-definite function $\Phi(y_1, \ldots, y_k, t)$ such that the inequality (1.6) is satisfied for all values of the variables t, x_j in the region (1.3) and for all the corresponding values of the variables t, y_s in the region (1.4), then the unperturbed motion (1.1) is stable with respect to the quantities $Q_1, \ldots, Q_k$.

Proof. [1]. According to the definition of a definite function we can find a positive-definite function $W(y_1, \ldots, y_k)$ independent of t, such that in the region (1.4) the inequality

$$\Phi(y_1, \ldots, y_k, t) \geqslant W(y, \ldots, y_k) \tag{1.7}$$

holds.

Let $A > 0$ be an arbitrarily small number, smaller than H_1, and let l be the exact lower bound of the function W on the sphere (A):

$$y_1^2 + \cdots + y_k^2 = A$$

The number l is obviously positive, since W represents a positive-definite function.

We shall examine the function $\phi(x_1, \ldots, x_{2n}, t_0)$; since it does not depend explicitly on the time, it can have an infinitely small upper bound; consequently, we can find for l values λ and λ_1 such that when the corresponding values of x_j and y_s satisfy the conditions $x_1^2 + \ldots + x_{2n}^2 < \lambda$ and $y_1^2 + \ldots + y_k^2 \leqslant \lambda$, respectively, the functions $\Phi(y_1, \ldots y_k, t_0)$ and $\phi_1(x_1, \ldots, x_{2n}, t_0)$ will satisfy the conditions

$$\Phi(y_1, \ldots, y_k, t_0) \leqslant \varphi(x_1, \ldots, x_{2n}, t_0) < l$$

When the initial values of the variables x_j, and the corresponding variables y_s are chosen to satisfy the inequality $x_{10}^2 + \ldots + x^2_{2n,0} < \lambda$, and the corresponding inequality $y_{10}^2 + \ldots + y^2_{k0} < \lambda_1$, then according to the conditions of the theorem we have the inequalities

$$W(y_1, \ldots, y_k) \leqslant \Phi(y_1, \ldots, y_k, t) \leqslant \varphi(x_1, \ldots, x_{2n}, t) < l \qquad (1.8)$$

We can conclude here, that the variables y_s satisfy the condition

$$y_1^2 + \cdots + y_k^2 < A$$

since l is the exact lower bound of the function W on the sphere (A). The theorem is proved.

(*Note*. If instead of (1.6) we had the inequality

$$\Phi(y_1,\ldots, y_k,t) \leqslant \varphi(x_1,\ldots,x_{2n},t) + C \qquad (C \geqslant 0)$$

then for a suitably chosen value of x_{i0} and for all values of the time the inequality $y_1^2 + \ldots + y_k^2 < A$ would be satisfied, where A is a number such that the exact lower bound of the function W on the sphere (A) is greater than the number C.

As an example we shall consider the well known problem of stability of rotation about a vertical axis of a heavy rigid body in the case of Lagrange [1].

Let p, q, r be the projections of the instantaneous angular velocity of a rigid body on its principal axes of inertia with respect to a fixed point; let γ, γ', γ'' be the direction cosines of the vertical axis with respect to the principal axes of inertia. The projections of the angular momentum on these axes are

$$G_1 = Ap, \quad G_2 = Aq, \quad G_3 = Cr$$

where A, C are the principal moments of inertia of the rigid body with respect to its fixed point.

We shall investigate the stability of rotation of the body about the vertical axis

$$p = q = 0, \qquad r = r_0, \qquad \gamma = \gamma' = 0, \qquad \gamma'' = 1 \qquad (1.9)$$

with respect to the quantities G_1, G_2, G_3, γ, γ', γ'' assuming that in the perturbed motion

$$G_3 = G + \varkappa, \qquad \gamma'' = 1 + \delta \qquad (G = Cr_0)$$

and retaining the previous symbols for the remaining variables.

On the strength of the obvious inequality

$$G_1^2 + G_2^2 + \varkappa^2 \leqslant D(Ap^2 + Aq^2 + C\zeta^2) \qquad (\varkappa = C\zeta)$$

where D is the greater of the two quantities A and C, the values of the function

$$\Phi \equiv \frac{1}{D}(G_1^2 + G_2^2 + \varkappa^2) + 2\lambda(G_1\gamma + G_2\gamma' + \varkappa\delta) - (mgz + G\lambda)(\gamma^2 + \gamma'^2 + \delta^2) \qquad (1.10)$$

for the perturbed motion are not greater than the values of the function

$$\varphi \equiv V_1 + 2\lambda V_2 - (mgz + G\lambda) V_3 - 2(G + C\lambda) V_4 = \text{const}$$

where $V_i (i = 1, \ldots, 4)$ are the first integrals of the equations of the perturbed motion (See [1] page 27). λ is a constant, mg is the weight of the body, z is the coordinate of the center of gravity.

According to the just-proved theorem the conditions for the positive-definiteness of the function (1.10) yield the sufficient conditions for stability of the unperturbed motion (1.9); this last condition could be reduced to the inequality

$$G^2 - 4Dmgz > 0 \qquad (1.11)$$

It is easily seen that if the above inequality is satisfied, then the Maievskii's condition $C^2 r_0^2 - 4\,Amgz > 0$ is also satisfied; it is well known that this last condition is the necessary and sufficient condition for the stability of (1.9)).

2. The theory given above could be useful in the application of the second Liapunov method to the problems of stability of motion of continuous media with respect to a finite number of parameters, which describe the motion through integrals, [2]. Such parameters could be, for example, the coordinates of the center of gravity of a bounded volume of a continuous medium, or projections of its linear momentum on certain axes, or similar quantities, whose variations with time are described by ordinary differential equations. The stability of motion of a continuous medium with respect to the above mentioned parameters will be called the conditional stability of motion of a continuous medium.

As an example we shall consider the problem of stability of rotation of a solid with a liquid-filled cavity, with respect to the parameters describing the motion of the solid and to the projections of the angular momentum of the liquid [2].

We shall consider a free solid with completely or partially liquid-filled cavity, and the liquid is assumed to be ideal, non-compressible and homogeneous. We shall also assume, for simplicity, that the central ellipsoid of inertia of the solid is an ellipsoid of revolution, and the cavity is also a body of revolution whose axis coincides with the axis of the ellipsoid. In the case when the liquid has a free surface we shall regard the pressure on it to be constant. We shall also assume that the motion of the liquid is continuous, and that the velocities of the liquid particles and the pressure are continuous functions of the coordinates.

Since among the possible displacements of the body and the liquid in its cavity are rotations about an arbitrary axis and also translatory displacements of the whole system, the solid plus the liquid, as a single rigid body, the theorem of the angular momentum of the system in its motion with respect to its center of mass, that is of the motion relative to the coordinate system $O_1x_1y_1z_1$ whose origin O_1 is at the common center of mass of the solid and the liquid, and whose axes are parallel to the fixed axes. In the problem of the stability of rotation of a solid with a liquid-filled cavity this circumstance permits the consideration only of the relative motion, as if the mass center O_1 of the system were fixed.

We shall introduce another rectangular coordinate system $Oxyz$, moving with the solid.

In the case when the liquid entirely fills the cavity, the origin of the moving coordinate system O will coincide with the mass center O_1 of the whole system and the coordinate axes will coincide with the principal axes of inertia of the solid. In the case, when the liquid in the cavity has a free surface under a constant pressure, the origin O will coincide with the mass center of the solid, and the coordinate axes will be along the principal central axes of inertia. In both cases the axis Oz will be along the common axis of revolution of the central ellipsoid of inertia of the solid and of its cavity. The moments of inertia of the solid about the x, y, z axes are $A = B$, C respectively, and the direction cosines of the constant direction axis O_1z_1 with respect to the fixed axes are γ_1, γ_2, γ_3 respectively.

In order to begin from some concrete example we shall consider the case when the center of mass of the whole system is in rectilinear motion with constant velocity. This case is the well known approximation to a small segment of a flat trajectory of a missile. We shall assume, as in the case of a solid propellant missile, that the liquid charged missile is subject only to the overturning couple of the forces of air pressure [3]. The moment of this couple is assumed to be proportional to the sine of the angle between the Oz axis and the direction of the velocity of the mas center of the system O_1; let the x, y, z components of this moment be $L_1 = a\gamma_2$, $L_2 = -a\gamma_1$, $L_3 = 0$ respectively, where a is a constant. We assume also that the axis O_1z_1 coincides with the velocity vector of the mass center O_1. The reasons for the above assumption will be omitted, but we shall mention that the smaller the volume of the air bubble in the filling liquid the better the approximation.

Applying the general theorems on relative motion of a mechanical system about its mass center we could obtain some of the first integrals of the equations of motion of a solid with a liquid-filled cavity.

When we consider the motion of the system with respect to the $O_1x_1y_1z_1$ axes, the real displacements of the body and of the liquid in its cavity belong to the set of possible displacements of the system. Since under the assumptions, listed above, the forces of the air pressure could be represented by the force function $U = -a\gamma_3$, the total kinetic energy in the relative motion of the system could be expressed as

$$T_1 + T_2 + a\gamma_3 = \mathrm{const} \tag{2.1}$$

where T_1 and T_2 represent the kinetic energies of the solid and of the liquid, respectively, in their motion with respect to the coordinate system $O_1x_1y_1z_1$.

Let v_1, v_2, v_3 and ω_1, ω_2, ω_3 be the x, y, z components of $\mathbf{V}_0$, which is the velocity vector of the point O, and of the instantaneous angular velocity vector $\bar{\omega}$ respectively. Then

$$2T_1 = M_1(v_1^2 + v_2^2 + v_3^2) + A(\omega_1^2 + \omega_2^2) + C\omega_3^2$$

where M_1 is the mass of the solid. In the case when the liquid fills the cavity entirely, it is obvious that $v_1 = v_2 = v_3 = 0$.

Let ρ be the density of the liquid and v_x, v_y, v_z be the x, y, z components of the velocity vector $\mathbf{V}$ of the particles of the liquid with respect to the coordinate system $O_1x_1y_1z_1$. Then

$$2T_2 = \rho\int_\tau (v_x^2 + v_y^2 + v_z^2)\, d\tau$$

where τ is the volume of the xyz space occupied by the liquid at the given instant.

Under our assumption the forces exerted on the body by air pressure have no torque about the O_1z_1 axis; therefore the relative motion of the system has the integral of areas

$$(A\omega_1 + g_1)\gamma_1 + (A\omega_2 + g_2)\gamma_2 + (C\omega_3 + g_3)\gamma_3 = \mathrm{const} \tag{2.2}$$

where by

$$g_1 = \rho\int_\tau (yv_z - zv_y)\, d\tau, \quad g_2 = \rho\int_\tau (zv_x - xv_z)\, d\tau, \quad g_3 = \rho\int_\tau (xv_y - yv_x)\, d\tau \tag{2.3}$$

are denoted the x, y, z components of the momentum of the liquid in its motion with respect to the coordinate axes $O_1x_1y_1z_1$.

Since the momentum of the relative motion of the system is

$$M_1\mathbf{v}_0 + \rho\int_\tau \mathbf{v}\, d\tau = 0$$

the angular momentum of the relative motion of the system is the same for all points of the space. We shall write down the following obvious identity

$$\gamma_1^2+\gamma_2^2+\gamma_3^2=1 \tag{2.4}$$

for the direction cosines of the O_1z_1 axis.

For cavities which are bodies of revolution, the moment of the pressure forces of an ideal liquid and of the air inside the cavity about the Oz axis, obviously equals zero. Since $A = B$ and $L_3 = 0$, at any instant of time during the motion, the z-component of the instantaneous angular velocity vector remains constant:

$$\omega_3=\text{const} \tag{2.5}$$

Among the real motions of a solid body with a cavity filled with an ideal liquid, we have a uniform rotation of the solid with the angular velocity ω about the Oz axis, which is colinear with the O_1z_1 axis; since in this case the motion of the liquid is steady and such that the x and y components of its angular momentum equal zero, and the z component is the constant g. Since according to our assumptions the liquid is inviscid, and the cavity is a body of revolution, this set of steady rotations includes in particular that of equilibrium with respect to the coordinate system $O_1x_1y_1z_1$; in such a case $g = 0$. Under certain conditions there is also possible another extreme case of the liquid rotating as a rigid body with the angular velocity ω. In this case

$$g=\omega\rho\int(x^2+y^2)d\tau$$

where τ_0 is the volume occupied by the liquid in this motion.

We shall consider now the stability of rotation of a solid and of the corresponding steady motion of the liquid in its cavity:

$$\begin{gathered}\omega_1=\omega_2=0, \qquad \omega_3=\omega, \qquad \gamma_1=\gamma_2=0, \qquad \gamma_3=1\\ v_1=v_2=v_3=0, \qquad g_1=g_2=0, \qquad g_3=g\end{gathered} \tag{2.6}$$

with respect to the quantities ω_1, ω_2, ω_3, γ_1, γ_2, γ_3, g_1, g_2, g_3, v_1, v_2, v_3. In the case when the liquid fills the cavity entirely we shall consider the stability with respect to the first nine of the quantities.

In the perturbed motion we shall substitute

$$\omega_3=\omega+\xi, \qquad g_3=g+\eta, \qquad \gamma_3=1+\zeta$$

and for the remaining variables we shall retain the previous symbols. The integrals (2.1), (2.2), (2.4), (2.5) for the perturbed motion become

$$V_1 = M_1 (v_1^2 + v_2^2 + v_3^2) + A (\omega_1^2 + \omega_2^2) + C (\xi^2 + 2\omega\xi) + 2T_2 + 2a\zeta = \text{const}$$

$$V_2 = (A\omega_1 + g_1)\gamma_1 + (A\omega_2 + g_2)\gamma_2 + C\xi + \eta + C(\omega + \xi)\zeta + (g + \eta)\zeta = \text{const}$$

$$V_3 = \gamma_1^2 + \gamma_2^2 + \zeta^2 + 2\zeta = 0, \qquad V_4 = \xi = \text{const} \tag{2.7}$$

We shall consider also the following function

$$H_1 \equiv M_1 (v_1^2 + v_2^2 + v_3^2) + A (\omega_1^2 + \omega_2^2) + C (\xi^2 + 2\omega\xi) + \frac{1}{S} (g_1^2 + g_2^2 + 2g\eta + \eta^2) + 2a\zeta$$

where S is a quantity proportional to the greatest of the principal moments of inertia of the liquid in its cavity for the point O.

On the strength of the Liapunov's inequality $g_1^2 + g_2^2 + g_3^2 < 2T_2S$ we can make the statement that

$$H_1 \leqslant V_1 = \text{const} \tag{2.8}$$

Let us consider the function

$$V = H_1 + 2\lambda V_2 - (a + C\omega\lambda + g\lambda) V_3 - 2C(\omega + \lambda) V_4 + \frac{C(C-A)}{A} V_4^2 =$$
$$= A\omega_1^2 + 2\lambda (A\omega_1 + g_1)\gamma_1 - (a + C\omega\lambda + g\lambda)\gamma_1^2 + \frac{1}{S} g_1^2 + M_1 v_1^2 +$$
$$+ A\omega_2^2 + 2\lambda (A\omega_2 + g_2)\gamma_2 - (a + C\omega\lambda + g\lambda)\gamma_2^2 + \frac{1}{S} g_2^2 + M_1 v_2^2 + \frac{C^2}{A}\xi^2 +$$
$$+ 2\lambda (C\xi + \eta)\zeta - (a + C\omega\lambda + g\lambda)\zeta^2 + \frac{1}{S}\eta^2 + M_1 v_3^2 + 2\left(\frac{g}{S} + \lambda\right)\eta \tag{2.9}$$

which represents the sum of three quadratic forms of the same type in four variables, and one linear function in the variable η ; λ is a constant. According to Sylvester's criterion, the necessary and sufficient condition that the quadratic part of V be positive-definite is that there exists a λ such that

$$(A + S)\lambda^2 + (C\omega + g)\lambda + a < 0 \tag{2.10}$$

The inequality is possible if the polynomial on the left hand side of (2.10) has two distinct real roots λ_1 and λ_2; that is if

$$(C\omega + g)^2 - 4(A + S)a > 0 \tag{2.11}$$

If the condition (2.11) is satisfied we could choose the constant λ arbitrarily in the interval $\lambda_1 < \lambda < \lambda_2$; if also

$$(g/S + \lambda)\eta \geqslant 0 \tag{2.12}$$

then the function V will be positive-definite in all its variables.

On the strength of the inequality (2.8)

$$V \leqslant V_1 + 2\lambda V_2 - (a + C\omega\lambda + g\lambda)V_3 - 2C(\omega + \lambda)V_4 + \frac{C(C-A)}{A}V_4^2$$

Consequently, when the conditions (2.11), (2.12) are satisfied the function V satisfies the conditions of the previously proved theorem.

Thus, the conditions (2.11), (2.12) could be regarded as the necessary conditions for the stability of the unperturbed motion of the system (2.6) with respect to the quantities ω_1, ω_2, ω_3, g_1, g_2, g_3, γ_1, γ_2, γ_3, v_1, v_2, v_3. Let us mention that if we set $\lambda = -g/S$, then the condition (2.12) will be satisfied, the inequality (2.10) assumes the form

$$\left(C\omega - A\frac{g}{S}\right)\frac{g}{S} - a > 0 \tag{2.13}$$

and becomes the single sufficient condition for stability of the unperturbed motion (2.6).

If we set $\lambda = -2a/C\omega$, then the inequalities (2.10) and (2.12) become the stability conditions (2.12) and (2.13) given in the paper [2].

It is also easily seen [2], that if the motion of the liquid entirely filling the cavity is always irrotational, with the velocity potential ϕ, then the condition (2.12) is satisfied.

We shall finally show that the condition (2.11) is the sufficient condition for the stability of the unperturbed motion (2.6) in the first approximation. Let us write down the third equation of the perturbed motion [2]:

$$\frac{d\eta}{dt} + \omega_1 g_2 - \omega_2 g_1 = 0$$

Regarding ω_1, ω_2, g_1, g_2 as small quantities of the first order and neglecting their products, we obtain the integral of the equation in variations

$$V_5 = \eta = \text{const}$$

We shall consider the function

$$W = V - 2\left(\frac{g}{S} + \lambda\right)V_5 \tag{2.14}$$

where the function V is defined by the equation (2.9). It is obvious, that the necessary and sufficient condition for the positive-definiteness of the function (2.14) is the condition (2.11) which proves out statement.

BIBLIOGRAPHY

1. Chetaev, N.G., *Ustoichivost dvizhenia* (*Stability of motion*). Gostekhizdat, 1955.

2. Rumiantsev, V.V., Ob ustoichivosti vrashchatolnykh dvizhenii tverdogo tela s zhidkim napolnenem (On the stability of rotary motions of a liquid filled solid). *PMM* Vol. 23, No. 6, 1959.

3. Chetaev, N.G., Ob ustoichivosti vrashchatelnykh dvizhenii tverdogo tela polost kotorogo napolnena idealnoi zhidkostiu (On the stability of rotary motions of a solid with a cavity filled with an ideal liquid). *PMM* Vol. 21, No. 2, 1957.

Translated by T.L.